Random Surfing on Multipartite Graphs

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Outline

Introduction & Motivation

PageRank Model

$$\mathbf{G} = \alpha\mathbf{H} + (1-\alpha)\mathbf{E}$$

Primitivity Adjustment of the Row-Normalized Adjacency Matrix $\mathbf{H}$:

- ***Damping Factor*** α
  - Has received much attention (Generalized Damping Functions (Functional Rankings) [1], Multidamping [5], ...)
- ***Teleportation matrix*** $\mathbf{E}$
  - Little have been done towards its generalization [8].

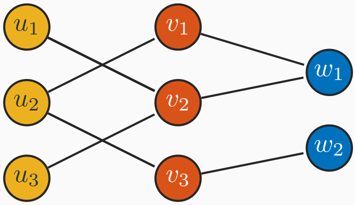
Revisiting the Random Surfer Model II

Problems With Traditional Teleportation

- Treats nodes in a **"leveling way"**
  - Restrictive or even counter-intuitive (eg. structured graphs)
- **Blind to the spectral characteristics** of the underlying graph

Overview of Our Approach

- We focus on **Multipartite Graphs**
- We modify the traditional teleportation model
  - ***Different Teleportation behavior per partite set.***



Block Teleportation Model: Definition, Algorithm and Theoretical Analysis

Model Definition

$$\mathbf{S} = \eta \mathbf{H} + \mu \mathbf{M}$$

$$\mathbf{H} \triangleq \mathrm{diag}(\mathbf{A}_{\mathcal{G}}\mathbf{1})^{-1}\mathbf{A}_{\mathcal{G}},$$

Each partite set is a **Teleportation Block!**

$$[\mathbf{M}]_{ij} \triangleq \begin{cases} \frac{1}{|\mathcal{M}_i|}, & \text{if } v_j \in \mathcal{M}_i, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{M}_i$ the origin partite set of v_i.

$$\mathbf{M} = \underbrace{\mathbf{R}\boldsymbol{\Delta}\mathbf{R}^{\intercal}}_{\text{Sparse and Low-Rank Factors}}$$

Random Surfing Interpretation

The Random Surfer:

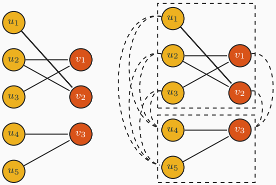
- With probability η follows the edges of the graph
- With probability $\mu \triangleq 1 - \eta$ he jumps to a node belonging to the same partite set he is currently in.

Decomposability and Time-Scale Dissociation

Theorem (Decomposability)

When the value of the teleportation parameter μ is small enough, the Markov chain corresponding to matrix $\mathbf{S}$ is NCD with respect to the partition of the nodes of the initial graph, into different connected components.

$$\mathbf{S} = \tilde{\mathbf{S}} + \varepsilon\mathbf{C}, \qquad \tilde{\mathbf{S}} \triangleq \operatorname{diag}\{\mathbf{S}(\mathcal{G}_1), \ldots, \mathbf{S}(\mathcal{G}_L)\}$$



$$\mathbf{S}^t = \underbrace{\mathbf{Z}_{1_1}}_{\text{Term A}} + \underbrace{\sum_{I=2}^{L} \lambda_{1_I}^t \mathbf{Z}_{1_I}}_{\text{Term B}} + \underbrace{\sum_{I=1}^{L}\sum_{m=2}^{n(I)} \lambda_{m_I}^t \mathbf{Z}_{m_I}}_{\text{Term C}},$$

$$\tilde{\mathbf{S}}^t = \underbrace{\sum_{I=1}^{L} \tilde{\mathbf{Z}}_{1_I}}_{\text{Term }\tilde{\text{A}}} + \underbrace{\sum_{I=1}^{L}\sum_{m=2}^{n(I)} (\tilde{\lambda})_{m_I}^t \tilde{\mathbf{Z}}_{m_I}}_{\text{Term }\tilde{\text{C}}}.$$

- **Short-term Dynamics.**
- **Short-term Equilibrium.**
- **Long-term Dynamics.**
- **Long-term Equilibrium.**

Computational Implications... in brush strokes!

Study each Connected Component in Isolation, and then combine the results.

BT-Rank Algorithm and Computational Analysis

Block-Teleportation Rank

Input: $\mathbf{H}, \mathbf{M} \in \Re^{n \times n}$, ϵ, scalars $\eta, \mu > 0$ such that $\eta + \mu = 1$.

Output: $\boldsymbol{\pi}^\intercal$

1: Let the initial approximation be $\boldsymbol{\pi}^\intercal_{(0)}$. Set $k = 0$.

2: Compute

$$\begin{aligned} \boldsymbol{\pi}^\intercal_{(k+1)} &\leftarrow \boldsymbol{\pi}^\intercal_{(k)} \mathbf{H} \\ \boldsymbol{\phi}^\intercal &\leftarrow \boldsymbol{\pi}^\intercal_{(k)} \mathbf{M} \\ \boldsymbol{\pi}^\intercal_{(k+1)} &\leftarrow \eta \boldsymbol{\pi}^\intercal_{(k+1)} + \mu \boldsymbol{\phi}^\intercal \end{aligned}$$

3: Normalize $\boldsymbol{\pi}^\intercal_{(k+1)}$ and compute $r = \|\boldsymbol{\pi}^\intercal_{(k+1)} - \boldsymbol{\pi}^\intercal_{(k)}\|_1$.

4: If $r < \epsilon$, quit with $\boldsymbol{\pi}^\intercal_{(k+1)}$, otherwise $k \leftarrow k + 1$ and go to step 2.

General Cost: $\boxed{\underbrace{\Theta(\mathrm{nnz}(\mathbf{H}))}_{\text{per iteration}} \frac{\log \epsilon}{\log |\lambda_2(\mathbf{S})|}}$

If $\chi(\mathcal{G}) = 2$ we can dig a little deeper!

Theorem (Eigenvalue Property)

Assuming $\mathcal{G}$ is a connected graph for which $\chi(\mathcal{G}) = 2$ holds, the spectrum of the stochastic matrix $\mathbf{S}$ is such that $-\eta + \mu \in \lambda(\mathbf{S})$.

Theorem (Lumpability)

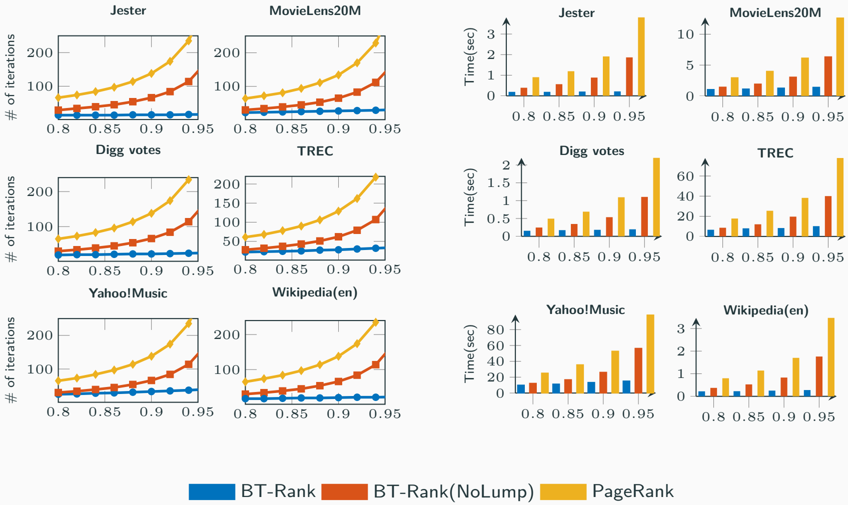
*The BT-Rank Markov chain that corresponds to a 2-chromatic graph, is **lumpable** wrt $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2\}$.*

Theorem (Perron Vector)

When the BT-Rank Markov chain is lumpable wrt to partition $\mathcal{A}$, for the left Perron eigenvector of matrix $\mathbf{S}$ it holds

$$\boldsymbol{\pi}_1^\intercal \mathbf{1}_{\mathcal{A}_1} = \boldsymbol{\pi}_2^\intercal \mathbf{1}_{\mathcal{A}_2}$$

Experimental Evaluation


Jester
MovieLens20M
Digg votes
TREC
Yahoo!Music
Wikipedia(en)
# of iterations
Time(sec)
0.8 0.85 0.9 0.95
BT-Rank
BT-Rank(NoLump)
PageRank

Qualitative Experiments: Ranking-Based Recommendation I

Our Approach

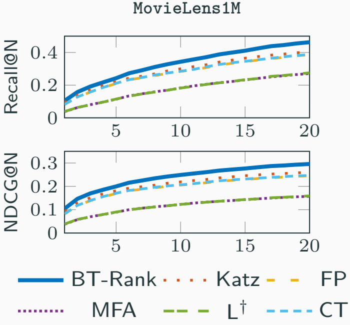
- We model the recommender as a tripartite graph

- Personalization through matrix $\mathbf{M} \triangleq \operatorname{diag}\left(\mathbf{1}\boldsymbol{e}_{\boldsymbol{i}}^{\intercal}, \mathbf{1}\boldsymbol{\omega}_{\boldsymbol{i}}^{\intercal}, \mathbf{1}\boldsymbol{\varpi}_{\boldsymbol{i}}^{\intercal}\right)$
  - $\boldsymbol{\omega}_i$: the normalized vector of the users' ratings over the set of movies.
  - $\boldsymbol{\varpi}_i$: the normalized vector of his mean ratings per genre.

Qualitative Experiments: Ranking-Based Recommendation II

Methodology

- Randomly sample 1.4% of the ratings $\Rightarrow$ probe set $\mathcal{P}$
- Use each item v_j, rated with 5 stars by user u_i in $\mathcal{P}$ $\Rightarrow$ test set $\mathcal{T}$
- Randomly select another 1000 unrated items of the same user for each item in $\mathcal{T}$
- Form ranked lists by ordering all the 1001 items



Metrics

- **Recall@N**
- **Normalized Discounted Cumulative Gain (NDCG@N)**
- **Mean Reciprocal Rank**

Conclusions & Future Work

Conclusion & Future Work

Synopsis

We proposed a simple alternative teleportation component for Random Surfing on Multipartite Graphs.

- Can be handled efficiently
- Entails nice theoretical properties
- Allows for richer modeling

Future Directions

- Propose a Systematic Framework for the definition of teleportation models that match better the underlying graphs
  - For the Web-Graph: **NCDawareRank** (WSDM'13)

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Questions?

Appendix: Eigenvalue Theorem

Back

Theorem (Eigenvalue Property)

Assuming $\mathcal{G}$ is a connected graph for which $\chi(\mathcal{G}) = 2$ holds, the spectrum of the stochastic matrix $\mathbf{S}$ is such that $-\eta + \mu \in \lambda(\mathbf{S})$.

Proof Sketch.

- We define a vector $\mathbf{v} \triangleq [\ \overbrace{1\ \ 1\ \ 1\ \ \cdots\ \ 1}^{\#\text{nodes of 1st partite set}}\quad \overbrace{-1\ \ -1\ \ \cdots\ \ -1}^{\#\text{nodes of 2nd partite set}}\ ]$
- We show that $(-1, \mathbf{v})$ is an eigenpair of matrix $\mathbf{H}$, and $(1, \mathbf{v})$ is an eigenpair of matrix $\mathbf{M}$.
- Then, using any nonsingular matrix, $\mathbf{Q} \triangleq \begin{pmatrix} \mathbf{1} & \mathbf{v} & \mathbf{X} \end{pmatrix}$, allows us to perform a similarity transformation

$$\begin{aligned} \mathbf{Q}^{-1}\mathbf{S}\mathbf{Q} &= \mathbf{Q}^{-1}\left(\eta\mathbf{H} + \mu\mathbf{M}\right)\mathbf{Q} = \cdots = \\ &= \begin{pmatrix} 1 & 0 & \eta\mathbf{y_1}^\intercal\mathbf{H}\mathbf{X} + \mu\mathbf{y_1}^\intercal\mathbf{M}\mathbf{X} \\ 0 & -\eta + \mu & \eta\mathbf{y_2}^\intercal\mathbf{H}\mathbf{X} + \mu\mathbf{y_2}^\intercal\mathbf{M}\mathbf{X} \\ \mathbf{0} & \mathbf{0} & \eta\mathbf{Y}^\intercal\mathbf{H}\mathbf{X} + \mu\mathbf{Y}^\intercal\mathbf{M}\mathbf{X} \end{pmatrix} \end{aligned} \tag{1}$$

that reveals the desired eigenvalue.

□

Appendix: Lumpability Theorem

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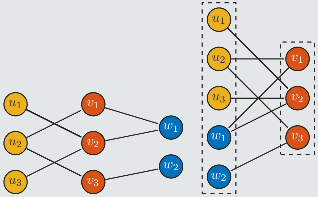
Theorem (Lumpability of the BT-Rank Chain)

The BT-Rank Markov chain that corresponds to a 2-chromatic graph, is ***lumpable****.*

Proof Sketch.



$$\mathbf{S} = \left(\begin{array}{ccccc:ccc}
\frac{\mu}{3} & \frac{\mu}{3} & \frac{\mu}{3} & & & & \eta & \\
\frac{\mu}{3} & \frac{\mu}{3} & \frac{\mu}{3} & & & \frac{\eta}{2} & & \frac{\eta}{2} \\
\frac{\mu}{3} & \frac{\mu}{3} & \frac{\mu}{3} & & & & \eta & \\
 & & & \frac{\mu}{2} & \frac{\mu}{2} & \frac{\eta}{2} & \frac{\eta}{2} & \\
 & & & \frac{\mu}{2} & \frac{\mu}{2} & & & \eta \\
\hdashline
 & \frac{\eta}{2} & & \frac{\eta}{2} & & \frac{\mu}{3} & \frac{\mu}{3} & \frac{\mu}{3} \\
 & & \frac{\eta}{2} & \frac{\eta}{2} & & \frac{\mu}{3} & \frac{\mu}{3} & \frac{\mu}{3} \\
 & \frac{\eta}{2} & & & \frac{\eta}{2} & \frac{\mu}{3} & \frac{\mu}{3} & \frac{\mu}{3}
\end{array}\right)$$

We have

- $\Pr\{i \to \mathcal{A}_2\} = \sum_{j \in \mathcal{A}_2} S_{ij} = \eta$ for all $i \in \mathcal{A}_1$.
- $\Pr\{i \to \mathcal{A}_1\} = \sum_{j \in \mathcal{A}_1} S_{ij} = \eta$ for all $i \in \mathcal{A}_2$.

$\square$

Appendix: Perron Vector Theorem

Back

Theorem (Eigenvector Property)

When the BT-Rank Markov chain is lumpable with respect to partition $\mathcal{A} = \{\mathcal{A}_1, \mathcal{A}_2\}$, for the left Perron-Frobenius eigenvector of matrix $\mathbf{S}$ it holds

$$\boldsymbol{\pi}_1^\intercal \mathbf{1}_{\mathcal{A}_1} = \boldsymbol{\pi}_2^\intercal \mathbf{1}_{\mathcal{A}_2}$$

Proof Sketch.

$$\begin{aligned}
\boldsymbol{\pi}^\intercal \mathbf{S} \begin{pmatrix} \mathbf{1}_{\mathcal{A}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{\mathcal{A}_2} \end{pmatrix} &= \boldsymbol{\pi}^\intercal \begin{pmatrix} \mathbf{1}_{\mathcal{A}_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{\mathcal{A}_2} \end{pmatrix} \\
\boldsymbol{\pi}^\intercal \begin{pmatrix} \mu \mathbf{M}_{11} \mathbf{1}_{\mathcal{A}_1} & \eta \mathbf{H}_{12} \mathbf{1}_{\mathcal{A}_2} \\ \eta \mathbf{H}_{21} \mathbf{1}_{\mathcal{A}_1} & \mu \mathbf{M}_{22} \mathbf{1}_{\mathcal{A}_2} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{\pi}_1^\intercal \mathbf{1}_{\mathcal{A}_1} & \boldsymbol{\pi}_2^\intercal \mathbf{1}_{\mathcal{A}_2} \end{pmatrix} \\
\begin{pmatrix} \boldsymbol{\pi}_1^\intercal & \boldsymbol{\pi}_2^\intercal \end{pmatrix} \begin{pmatrix} \mu \mathbf{1}_{\mathcal{A}_1} & \eta \mathbf{1}_{\mathcal{A}_1} \\ \eta \mathbf{1}_{\mathcal{A}_2} & \mu \mathbf{1}_{\mathcal{A}_2} \end{pmatrix} &= \begin{pmatrix} \boldsymbol{\pi}_1^\intercal \mathbf{1}_{\mathcal{A}_1} & \boldsymbol{\pi}_2^\intercal \mathbf{1}_{\mathcal{A}_2} \end{pmatrix}
\end{aligned}$$

and the result follows from the solution of the system. $\square$

Appendix: Near Complete Decomposability

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Herbert A. Simon

Sparsity $\longleftrightarrow$ Hierarchy $\longleftrightarrow$ Decomposability [10].

Nearly Completely Decomposable Systems

Interactions: Strong Within Blocks - Weak Between Blocks

- Successful Applications in Diverse Disciplines
  - Economics, Cognitive Theory, Management, Biology, Ecology, etc.
- Computer Science and Engineering:
  - Performance Analysis (Courtois [2])
  - Web Search (NG13 [8])
  - Recommendation and Data Mining (NG14 [7])